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1985 J. Phys. A: Math. Gen. 18 L975

(http://iopscience.iop.org/0305-4470/18/15/013)

J. Phys. A: Math. Gen. 18 (1985) L975–L978. Printed in Great Britain

LETTER TO THE EDITOR

Supersymmetric composition of interactions

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Received 6 August 1985

Abstract. The solution of the interaction composition problem found in the author's preceding work is extended to the case of simple supersymmetry.

In our preceding work (Lev 1984) the problem of interaction composition in three basic forms of relativistic dynamics proposed by Dirac (1949) has been discussed in detail. We also showed that in the instant form there exists a simple solution of the problem, and this solution agrees with that obtained by Coester and Polyzou (1982) within the framework of the multichannel scattering theory. An analogous solution was also found by Mutze (1984). Note that although in their works Coester and Polyzou (1982), Mutze (1984) and Lev (1984) used different techniques, the basis of these works is the Sokolov method of packing operators (Sokolov 1977, 1978).

In this work we show that the scheme used (Lev 1984) can be extended to a case when the theory is not only Poincaré, but super-Poincaré-invariant. For simplicity, we limit ourselves to the case of simple $N=1$ supersymmetry. As will become evident hereinafter, in the case of extended supersymmetry there may arise considerable technical complications; however, the basic principles of the problem concerning the transition from conventional symmetry to supersymmetry are not changed.

Let $x \to \hat{T}(x)$ be some representation of the Poincaré superalgebra in Hilbert space $\mathscr{H}$. Suppose that this representation is a description of the considered system in the instant form, i.e. operators of the 3-momentum $\boldsymbol{P}$ and representation generators of the group SU(2) are free of interaction. As in our previous work (Lev 1984), over the representation $x \to \hat{T}(x)$ there may be constructed a decomposition $\mathscr{H} = \int \oplus \mathscr{H}(\boldsymbol{p}) \mathrm{d}^3\boldsymbol{p}$, and the spin operators $\boldsymbol{j}$ acting in $\mathscr{H}(0)$ can be defined, where $\mathscr{H}(0) = \mathscr{H}(\boldsymbol{p})$ at $\boldsymbol{p}=0$. It follows from the von Neumann theorem that the mass operator $\hat{M}$, the parity operator (in terms of Z_2 grading) and the representation operators of odd elements of Poincaré superalgebra $\hat{Q}^\alpha (\alpha = 1, 2, 3, 4)$ are decomposable operators in the representation $\mathscr{H} = \int \oplus \mathscr{H}(\boldsymbol{p}) \mathrm{d}^3\boldsymbol{p}$. Let us denote through $\hat{m}$ and $\hat{q}^\alpha$ the reduction on $\mathscr{H}(0)$ of operators $\hat{M}$ and $\hat{Q}^\alpha$ respectively.

A set of operators $\hat{Q}^\alpha$ forms the Majorana spinor which satisfies, in particular, the following commutation and anticommutation relations

$$[\hat{Q}, \hat{M}_{\mu\nu}] = -\mathrm{i}\sigma_{\mu\nu}\hat{Q} \qquad \{\hat{Q}^\alpha, \hat{Q}^\beta\} = \tfrac{1}{2}(\gamma^\mu C)^{\alpha\beta}\hat{P}_\mu \tag{1}$$

where $\mu, \nu = 0,1,2,3$, $\hat{M}_{\mu\nu}$ are representation generators of the Lorentz group, $\hat{P}_\mu$-operators of 4-momentum and $\sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$. In the Majorana representation, matrices

γ_μ can be represented as direct products (see e.g. Novozhilov 1972):

$$\gamma^0=\rho_2\times\sigma_1 \qquad \gamma^1=\mathrm{i}\rho_3\times 1 \qquad \gamma^2=\mathrm{i}\rho_2\times\sigma_2 \qquad \gamma^3=-\mathrm{i}\rho_1\times 1 \tag{2}$$

where $\{\boldsymbol{\rho}\}$ and $\{\boldsymbol{\sigma}\}$ are two sets of Pauli matrices. In the Majorana representation, $C=\pm\gamma^0$. We choose the sign '+' to provide for positivity of energy.

As in the previous work (Lev 1984), one can construct a unitary operator $\hat{\mathcal{U}}^{-1}=\int\oplus\,\hat{\mathcal{U}}(\boldsymbol{p})^{-1}\,\mathrm{d}^3\boldsymbol{p}$ from $\mathcal{H}$ to $L_2(\boldsymbol{p})\otimes\mathcal{H}(0)$, where $L_2(\boldsymbol{p})$ is the space of complex functions of $\boldsymbol{p}$, such that $\int|f(\boldsymbol{p})|^2\,\mathrm{d}^3\boldsymbol{p}<\infty$. In complete analogy with the case of one-particle representations one can directly calculate an explicit form of representation operators of Poincaré superalgebra in the space $L_2(\boldsymbol{p})\otimes\mathcal{H}(0)$. The representation operators of even elements of Poincaré superalgebra are of the same form as those in the Poincaré-invariant theory (see, e.g., Lev 1984), and operators $\hat{Q}$ are of the following form

$$\hat{Q}=\int\exp[\tfrac{1}{2}\omega^{\mu\nu}(p/\lambda)\sigma_{\mu\nu}]\,\mathrm{d}\hat{e}(\lambda)\hat{q} \tag{3}$$

where $\omega^{\mu\nu}(\boldsymbol{p}/\lambda)$ are the parameters of a purely Lorentz boost defined by the 3-vector $\boldsymbol{p}/\lambda$, $\hat{e}(\lambda)$ the spectral function of the operator $\hat{m}$, and the integral in formula (3) is understood as a strong limit of the corresponding Riemannian sums. Operators $\hat{m}$, $\boldsymbol{j}$ and $\hat{q}$ acting in $\mathcal{H}(0)$ satisfy the commutation–anticommutation relations the form of which is easily derived from the form of corresponding relations in the representation of Poincaré superalgebra.

Let $\hat{m}^{1/2}$ denote a positive square root of the operator m. Then the operators

$$\hat{C}_1=(\hat{q}^1+\mathrm{i}\hat{q}^3)\hat{m}^{-1/2}, \qquad \hat{C}_2=(\hat{q}^2-\mathrm{i}\hat{q}^4)\hat{m}^{-1/2} \tag{4}$$

satisfy the anticommutation relations for the creation and destruction operators: $\{\hat{C}_1,\hat{C}_1^*\}=\{\hat{C}_2,\hat{C}_2^*\}=1$, and other anticommutators are equal to zero. The symbol '*' corresponds to the conventional Hermitian conjugation, since we consider the representation of Poincaré superalgebra in which the operators $\hat{Q}$ are Hermitian. Proceeding from the formulae for commutation of operators $\boldsymbol{j}$ and $\hat{q}$ and from (2), one can easily obtain through a direct calculation that

$$[\boldsymbol{j},\hat{C}_1]=\tfrac{1}{2}(-\mathrm{i}\hat{C}_2,-\hat{C}_1,-\hat{C}_2) \qquad [\boldsymbol{j},\hat{C}_2]=\tfrac{1}{2}(\mathrm{i}\hat{C}_1,\hat{C}_2,-\hat{C}_1). \tag{5}$$

Let us now proceed to the direct solution of the problem of interaction composition (see Lev 1984). Let a denote some partition of the considered system into subsystems $a_1\ldots a_n$, and $c(a)$ a set of operators $\{\hat{C}_1,\hat{C}_1^*,\hat{C}_2,\hat{C}_2^*\}$ for such partition. We denote by $m(a)$, $\mathcal{U}(a)$, etc the operators $\hat{m}$, $\hat{\mathcal{U}}$, etc for such partition. Since we consider a description in the instant form, then for any partition a $\boldsymbol{j}(a)=\boldsymbol{j}$, $\boldsymbol{P}(a)=\boldsymbol{P}$. We denote by c, $\mathcal{U}$, etc the operators $c(a)$, $\mathcal{U}(a)$, etc for a case when all interactions in our system are eliminated. Let $A(0)$ be a unitary operator in $\mathcal{H}(0)$, which commutes with the parity operator in $\mathcal{H}(0)$ and satisfies the conditions:

$$A(a)\boldsymbol{j}A(a)^{-1}=\boldsymbol{j}, \qquad A(a)cA(a)^{-1}=c(a). \tag{6}$$

We denote

$$\mathcal{A}(a)=\int\oplus\,\mathcal{U}(\boldsymbol{p};a)A(a)\mathcal{U}(\boldsymbol{p})^{-1}\,\mathrm{d}^3\boldsymbol{p}$$

$$\tilde{m}(a)=A(a)^{-1}m(a)A(a). \tag{7}$$

According to the Sokolov method of packing operators (1977, 1978) (see also Coester and Polyzou 1982, Mutze 1984, Lev 1984), to solve the problem of interaction composition, one should construct a unitary operator $\mathscr{A}$ from operators $\mathscr{A}(a)$ at different partitions a and an Hermitian operator $\tilde{m}$ from operators $\tilde{m}(a)$ in such a way that upon elimination of interactions corresponding to an arbitrary partition b the operator $\mathscr{A}$ goes to $\mathscr{A}(b)$ and the operator $\tilde{m}$ to $\tilde{m}(b)$. A solution of this combinatorial problem was given by Sokolov (1977) (see also Coester and Polyzou 1982, Mutze 1984), but we shall not apply it to the corresponding formulae. We just note that the symmetry property of operators $\mathscr{A}(a)$ and $\tilde{m}(a)$ is rather important. This property means that the operators $\mathscr{A}(a)$ and $\tilde{m}(a)$ should not depend on the order in which interaction has been eliminated between subsystems $a_1 \ldots a_n$ (for a detailed description of the symmetry property see, e.g., Lev 1983). In the work by Coester and Polyzou (1982) the symmetry property is formulated in the form $\mathscr{A}(a)_b = \mathscr{A}(a \cap b)$ and in an analogous way for $\tilde{m}(a)$. These properties are satisfied if the operator $A(a)$ also satisfies the symmetry condition (see Lev 1984).

The space $\mathscr{H}(0)$ can be represented as a direct orthogonal sum of four spaces

$$\mathscr{H}(0) = \mathscr{H}_0(a) \oplus \mathscr{H}_1(a) \oplus \mathscr{H}_2(a) \oplus \mathscr{H}_3(a) \tag{8}$$

where $\mathscr{H}_0(a)$ comprises the vectors annulled by operators $c_1(a)$ and $c_2(a)$, $\mathscr{H}_1(a) = c_1(a)^* \mathscr{H}_0(a)$, $\mathscr{H}_2(a) = c_2(a)^* \mathscr{H}_0(a)$, $\mathscr{H}_3(a) = c_1(a)^* c_2(a)^* \mathscr{H}_0(a)$. We denote projectors onto these subspaces as $\Pi_i(a)$, their explicit expressions being

$$\Pi_0(a) = c_2(a)c_1(a)c_1(a)^*c_2(a)^* \qquad \Pi_1(a) = c_1(a)^*c_2(a)c_2(a)^*c_1(a)$$
$$\Pi_2(a) = c_2(a)^*c_1(a)c_1(a)^*c_2(a) \qquad \Pi_3(a) = c_1(a)^*c_2(a)^*c_2(a)c_1(a). \tag{9}$$

Let $B(a)$ be an operator which isometrically maps $\mathscr{H}_0$ onto $\mathscr{H}_0(a)$ and commutes with the parity operator in $\mathscr{H}(0)$. If this operator satisfies the symmetry property and commutes with j, then, considering also the formulae (5) and (9), one can easily confirm that the solution for $A(a)$ is of a form

$$\begin{aligned} A(a) = {} & \Pi_0(a)B(a)\Pi_0 + \Pi_1(a)c_1(a)^*B(a)c_1\Pi_1 \\ & + \Pi_2(a)c_2(a)^*B(a)c_2\Pi_2 + \Pi_3(a)c_1(a)^*c_2(a)^*B(a)c_2c_1\Pi_3. \end{aligned} \tag{10}$$

Thus, one should find only the operator $B(a)$. Since operators Π_0 and $\Pi_0(a)$ commute with j and with the parity operator in $\mathscr{H}(0)$ (which follows from (5) and (9)), then such an operator $B(a)$ which isometrically maps $\mathscr{H}_0$ onto $\mathscr{H}_0(a)$ and is expressed only through operators Π_0 and $\Pi_0(a)$ satisfies all conditions. It is well known that such an operator exists if the following condition is satisfied:

$$\|\Pi_0(a) - \Pi_0\| < 1. \tag{11}$$

We denote $R(a) = (\Pi_0(a) - \Pi_0)^2$. One can directly confirm that $R(a)$ commutes with $\Pi_0(a)$ and Π_0, and the operator $B(a)$ at the condition (11) can be defined by the formula (see, e.g., Kato 1966)

$$B(a) = \Pi_0(a)\Pi_0(1 - R(a))^{-1/2}. \tag{12}$$

Thus, if the condition (11) is satisfied, the operator $A(a)$ satisfying all requirements is given by (10) and (12). The condition (11) is thought to be a natural one in terms of the general perturbation theory of linear operators (see, e.g., Kato 1966); however, its verification can be accomplished only in specific models.

As noted in the work by Lev (1984), in the case of conventional relativistic invariance there exists the solution $A(a)=1$, at all a, in the instant form, whereas in other forms the 'packing operators' $A(a)$ are necessarily non-trivial. We see now that in the supercase the 'packing' is non-trivial in the instant form as well. This is due to the fact that operators $m(a)$ commute with $q(a)$ which at different a, generally speaking, differ from each other, and all operators $\tilde{m}(a)$ should commute with the free operators q.

The scheme presented can be used for various applications. In particular, in an analogy with the case of conventional relativistic invariance (see, e.g., Lev 1985) one can construct the relativistic quantum mechanics of superparticles.

The author is grateful to L A Kondratyuk and S N Sokolov for useful discussions, as well as to G V Blankov for translating and typing the manuscript.

References

Coester F and Polyzou W N 1982 *Phys. Rev.* D **26** 1348-67
Dirac P A M 1949 *Rev. Mod. Phys.* **21** 392-9
Kato T 1966 *Perturbation Theory for Linear Operators* (Berlin: Springer)
Lev F M 1983 *Yad. Fiz.* **37** 1058-69
—— 1984 *J. Phys. A: Math. Gen.* **17** 2047-58
—— 1985 *Nucl. Phys.* A **433** 605-18
Mutze U 1984 *Phys. Rev.* D **29** 2255-69
Novozhilov Yu V 1972 *Introduction into the Theory of Elementary Particles* (in Russian) (Moscow: Nauka)
Sokolov S N 1977 *Dokl. Akad. Nauk* **233** 575-9
—— 1978 *Teor. Mat. Fiz.* **36** 193-207